The Relationship Between the Aldrich-McKelvey Scaling Solution and the Individual Differences Problem

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In the individual differences problem, the individuals report judged similarities/dissimilarities between objects/stimuli. Let \( d^2_{ijm} \) be the reported dissimilarity by individual \( i (i=1,\ldots,p) \) between stimulus \( j \) and stimulus \( m (j,m=1,\ldots,q) \) and let \( z_{jk} \) be the true location of the \( j \)th stimulus on the \( k \)th dimension \((k=1,\ldots,s)\). The reported dissimilarity is assumed to be the sum of the true dissimilarity and a noise term:

\[
d^2_{ijm} = \sum_{k=1}^{s} w_{ik} (z_{jk} - z_{jm})^2 + u_{ijm}
\]

(1)

where \( w_{ik} \) is the weight individual \( i \) places on dimension \( k \) (hence the term "individual differences").

In the classic Carroll-Chang (1970) solution for the individual differences problem, the \( p \times q \) by \( q \times q \) matrices of dissimilarites (taken as squared distances) are double-centered and an alternating least squares procedure is used to estimate the \( q \) by \( s \) matrix \( Z \) and the \( p \times q \) by \( q \) diagonal matrices \( W_i \).

The purpose of this paper is to show an alternative solution for the individual differences problem using a method developed by Aldrich and McKelvey (1977) for scaling individuals’ reported perceptions of the locations of stimuli along a scale with labeled endpoints.

In particular, let \( z_{ji} \) be the perceived location of stimulus \( j \) by individual \( i \). Aldrich and McKelvey assume that the individual reports a noisy linear transformation of the true location of the stimulus; that is
\[ \alpha_i + \beta_i z_{ji} = z_j + u_{ij} \]  

(2)

where \( u_{ij} \) satisfies the usual Gauss-Markov assumptions of zero mean, constant variance across stimuli and individuals, and zero covariance.

Let \( \hat{z}_j \) be the estimated location of stimulus \( j \) and let \( \hat{\alpha}_i \) and \( \hat{\beta}_i \) be the estimates of \( \alpha \) and \( \beta \).

Define

\[ e_i = \hat{\alpha}_i + \hat{\beta}_i z_{ji} - \hat{z}_j \]  

(3)

Aldrich and McKelvey set up the following Lagrangean multiplier problem:

\[ L(\alpha, \beta, z, \lambda_1, \lambda_2) = \sum_{i=1}^{n} e_i^2 + 2\lambda_1 \sum_{j=1}^{q} \hat{z}_j + \lambda_2 \left[ \sum_{j=1}^{q} \hat{z}_j^2 - 1 \right] \]  

(4)

that is, minimize the sum of squared error subject to the constraints that the estimated stimuli coordinates have zero mean and sum of squares equal to one. Define the \( q \) by 2 matrix \( X \) as

\[
X_i = \begin{bmatrix}
1 & z_{1i} \\
1 & z_{2i} \\
. & . \\
. & . \\
1 & z_{qi}
\end{bmatrix}
\]

then the solution for the individual transformations is simply the "least-squares regression of the reported on the actual (unknown) positions of the candidates" (Aldrich and McKelvey, p. 115).

That is,

\[
\begin{bmatrix}
\hat{\alpha}_i \\
\hat{\beta}_i
\end{bmatrix} = [X_i'X_i]^{-1} X_i'\hat{z} 
\]

(5)

where \( \hat{z} \) is the \( q \) by 1 vector of the actual (unknown) positions of the candidates.
To get the solution for $\hat{z}$, define the $q$ by $q$ matrix $A$ as:

$$A = \left[ \sum_{i=1}^{p} X_i (X_i' X_i)^{-1} X_i' \right]$$

Aldrich and McKelvey show that the partial derivatives for the $\hat{z}_j$ can be rearranged into the linear system:

$$\left[ A - p I_q \right] \hat{z} = \lambda_2 \hat{z}$$  \hspace{1cm} (6)

where $I_q$ is the $q$ by $q$ identity matrix. By equation (6), $\hat{z}$ is simply an eigenvector of the matrix $(A - p I_q)$ and $\lambda_2$ is the corresponding eigenvalue.

To determine which of the $q$ possible eigenvectors is the solution, Aldrich and McKelvey show that

$$-\lambda_2 = \sum_{i=1}^{p} e_i^2 = -\hat{z}' \left[ A - p I_q \right] \hat{z}$$ \hspace{1cm} (7)

Hence, the solution is the eigenvector of $(A - p I_q)$ "with the highest (negative) nonzero" eigenvalue. The solution for $\hat{z}$ from (6) can be taken back to equation (5) to solve for the individual transformation parameters.

Aldrich and McKelvey's ingenious solution is also a solution for the one dimensional individual differences problem. To see this, regard the reported dissimilarites from the individuals as responses on a scale labeled "zero dissimilarity" at one end and "very large dissimilarity" at the
other end. The $d_{ij}^2$ (or $d_{ijm}$) from equation (1) play the role of $z_{ji}$ in equation (2). In this framework, let $q^*$ be the number of stimuli. Then there are $q = q^*(q^*-1)/2$ $z_{ji}$'s for each individual. The $q^*(q^*-1)/2$ unique elements of the dissimilarities matrix gathered from each individual are placed in $X_i$. Therefore, $\hat{z}$ from equation (6) is of length $q^*(q^*-1)/2$ and is a noisy linear transformation of the true dissimilarities; that is

$$a + b\hat{z} = \Delta + u$$

where $a$ and $b$ are scalars, $\Delta$ is a $q^*(q^*-1)/2$ length vector of the true dissimilarities, and $u$ is a $q^*(q^*-1)/2$ length vector of error terms. Note that $\hat{z}$ will not necessarily satisfy the triangle inequality and that it does not matter whether or not the data gathered from the individuals is treated as distances or squared distances. To solve for $\Delta$, simply take the estimated $\hat{\alpha}_i$ and $\hat{\beta}_i$, solve for the "true" dissimilarities for each individual, and take the mean over the individuals as the estimator for $\Delta$: that is:

$$\hat{d}_{jm} = \left[ \frac{\sum_{i=1}^{p} (\hat{z}_{jm} - \hat{\alpha}_i)}{\hat{\beta}_i} \right]$$

Again, note that if the original data is regarded as squared distances, then equation (9) would be the formula for $\hat{d}_{jm}^2$. In any event, it is a simple matter to double-center the matrix of $\hat{d}_{jm}^2$ and obtain the estimate of the stimulus coordinates.

A nice feature of this approach is that it also generalizes to the s-dimensional case in which the individuals are constrained to have the same ratio of weights over the dimensions. That is, for individuals $i$ and $h$: $\frac{w_{i1}}{w_{i2}} = \frac{w_{h1}}{w_{h2}}$, and so on.
References
